Extraction of coherent vortices from homogeneous turbulence using curvelets and total variation filtering methods

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ABSTRACT

Most filtering methods for turbulent flow depend on wavelet and Fourier decompositions; however, this paper employs curvelets and total variation (TV) filtering methods. These two methods are investigated to extract the coherent and incoherent parts of a forced homogeneous isotropic turbulent field. The Lattice Boltzmann method with resolutions of 128³ and 256³ is used to simulate the turbulent fields, and the Q-identification method is applied to extract the elongated vortical structures. Most of the previous efforts apply filtering techniques to the velocity or vorticity fields; however, this paper applies the two filtering methods to the Q-field. 3D curvelet filtering is applied using a nonlinear thresholding technique. The 3D total variation method is applied using the split Bregman regularization method. The results indicate that the two filtering methods identify the coherent and incoherent parts smoothly. The results of the two filtering methods tend to identify coherent vortices and remove incoherent noise without any change in the physical structure of the turbulent fields. The results are compared with previous efforts using wavelet and Fourier decompositions.

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1. Introduction

In recent years, pure and applied mathematics have been the basis for filtering 2D and 3D data. The different mathematical techniques play an important role in filtering theory. Some of these filtering techniques borrow ideas from partial differential equations, such as the level set method [1,2] and nonlinear diffusion [3,4]. Other techniques exploit the decomposition of the experimental data into the wavelet domain to eliminate noise in the data [5,6]. Some other researchers have used different basis functions such as curvelets for image filtering [7–9]. A different class of methods depends on applying nonlinear optimization methods such as the total variation method [10,11]. Recently, these filtering methods have attracted the attention of researchers in fluid mechanics for studying the characteristics of turbulent flow. The principle of filtering in turbulent flow consists of separating the coherent flow from the noise parts, which are supposed to be Gaussian and distributed independently [12]. The popular filtering methods in turbulent flow depend on wavelet analysis [13–16].

In this paper, two other techniques, namely the curvelet and total variation filtering methods, are used to decompose the flow into coherent and incoherent parts. The curvelets and total variation filtering methods are popular techniques in the area of image processing, especially for eliminating noise. This paper demonstrates that these techniques are also effective for extracting coherent vortices from a turbulent flow field. The filtering techniques are commonly applied to the velocity or vorticity vector fields; however, in this study, the two filtering methods are applied to the scalar Q-field. Conventional vortex-identifying methods such as the Q method are effective for defining vortices, but no established method can distinguish noise from extracted structures. A number of tiny structures including noise also appear as the threshold approaches zero. Thus, if the value of Q is used to extract coherent vortices, a proper noise filter is required. The present paper examines two filters, curvelet and TV filters, and discusses their noise reduction effects. Dealing with this scalar field reduces the time cost and minimizes the computational error compared with a vector field. Also, based on the definition of Q regions with high Q values representing vortical structures, the curvelets and total variation methods identify regions with high rotation values as the coherent part. The incoherent part corresponds to the values of Q lower than a certain Q threshold.

To preserve the major features of the turbulent field, a statistical method is required for thresholding. In curvelet filtering, the threshold was chosen by using the statistical distribution of the noise that can be obtained in the frequency domain. In the TV filtering method, a minimization technique was used to optimize

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noise identification. A simple threshold is difficult to determine without applying any filtering techniques. There is no guarantee that this simple threshold removes only the incoherent part without removing coherent structures. The noise is distributed everywhere throughout the field, and it is difficult to find a simple threshold to remove it. The curvelet transform is a multiscale directional transform that allows an almost optimal non-adaptive sparse representation of objects with edges. Other filtering techniques based on the wavelet domain ignore the geometric properties of structures and do not exploit the regularity of edges. Also, one of the main advantages of the TV technique is that it takes the filtering analysis into a continuous domain, simplifying the formalization of the model, which becomes independent from the grid used in the discrete problem. Recently, the curvelets and TV filtering methods have attracted many researchers as new tools for filtering two- and three-dimensional turbulent flows. Morino and Pullin [17] developed a multi-scale methodology for studying the non-local geometry of eddy structures in turbulence. In their study, a multi-scale decomposition based on the curvelet transform is applied to the full three-dimensional field, resulting in a finite set of component three-dimensional fields. Ma et al. [18] briefly review curvelets and their relation to classical wavelet transforms and multiscale geometric analysis, and this is systematically applied to turbulent flows in two and three dimensions. Constrained multiscale minimization results in minimum loss of geometric flow features and the extraction of coherent structures with their edges and geometry properly preserved, which is significant for turbulence modeling. The effectiveness of the curvelets analysis as compared to the wavelet transform has been demonstrated for both two- and three-dimensional turbulent flows. Many researchers such as Vincent and Meneguzzi [19] and Kaneda et al. [20] studied direct numerical simulations of homogeneous isotropic turbulence. In their studies, they found that the energy spectrum displays an inertial sub-range, and their visualization of the flow confirmed that the strongest vorticity is organized in very elongated thin tubes. Their simulations were performed using the spectral method to solve the Navier–Stokes equations. These studies suggest that the energy spectrum in the inertial sub-range almost follows the $k^{-5/3}$ Kolmogorov scaling law, where $k$ is the wave number.

Recently, the Lattice Boltzmann method (LBM) has been used by many researchers as a new computational tool to simulate turbulent flow. Decaying homogeneous isotropic turbulence Yu et al. [21] has been the focus of such simulations. To our knowledge, few studies have investigated forced homogeneous turbulence [22,23]. Cate et al. [23] used spectral forcing with LBM, in which they used the Fourier transformation for the forcing term in each time step.

This paper is organized as follows. Section 2 discusses the Lattice Boltzmann Method and the characteristics of the experimental turbulent flow data. The curvelets and the total variation filtering methods are discussed in Sections 3 and 4. Section 5 is devoted to the results of the study and a discussion of the obtained turbulent features. Section 6 summarizes our conclusions.

2. Characteristics of the numerical data

In this study, the computation domains are periodic boxes of lengths $L_x = 128^3$ and $L_z = 256^3$. The D3Q19 model is used to simulate the turbulent field with a resolution of $128^3$; the D3Q15 lattice model is used to generate the $256^3$ velocity field. Previous efforts investigated and simulated many fluid dynamic problems using the Lattice Boltzmann method. The LBM and its connection with the Navier–Stokes equation were reviewed by Chen and Doolen [24]. Ubertini and Succi [25] and Qi [26] discussed the recent advances in Lattice Boltzmann techniques for fluid engineering problems.

The Bhatnagar–Gross–Krook (BGK) [27] single relaxation time approximation used in the Lattice Boltzmann equation is defined as

$$f_s(x + v \Delta t, t + \Delta t) = \frac{1}{\tau} \left[ f_s(x, t) - f_{eq}^s(x, t) \right] + \frac{3}{2} \rho \left[ \mathbf{u}_s \cdot \nabla f(x, t) \right],$$

where $\tau$ is the single relaxation time and $3 \rho \mathbf{u}_s \cdot \nabla f(x, t)$ represents an additional forcing term to the Boltzmann equation (see Succi [28] and Casgrove et al. [29]). This implementation of the forcing function satisfies the continuity and Navier–Stokes equations up to the second order. The equilibrium distribution function $f_{eq}^s(x, t)$ is given by

$$f_{eq}^s(x, t) = \rho \left[ 1 + \frac{3}{2} \mathbf{u}_s \cdot \mathbf{u} - \frac{3}{2} |\mathbf{u}|^2 \right],$$

where $\rho$ is the fluid mass density and $\mathbf{u}$ is the velocity. The weighting coefficients $w_s$ depend on the discrete velocity set $\mathbf{e}_s$ and the dimensions of space. This study uses the 19-velocity LBM model on a 3D-cubic lattice. This paper will introduce the discrete velocity set and the weighting coefficients for the D3Q19 model only. The discrete velocity set is defined by

$$\mathbf{e}_s = \left\{ \left( \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \right), \left( \frac{3}{4}, 0, \frac{3}{4} \right), \left( 1, \frac{3}{4}, \frac{3}{4} \right) \right\}$$

The weighting coefficients $w_s$ are

$$w_s = \begin{cases} \frac{3}{10}, & \mathbf{e}_s = \left( \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \right) \\ \frac{3}{10}, & \mathbf{e}_s = \left( \frac{3}{4}, 0, \frac{3}{4} \right) \\ \frac{3}{10}, & \mathbf{e}_s = \left( 1, \frac{3}{4}, \frac{3}{4} \right) \end{cases}$$

More details on the simulation methods can be found in [30]. The LBM can simulate the Navier–Stokes equations with the viscosity $v = \frac{1}{15} (\tau - \frac{1}{3}) \mathbf{x}$, where $\mathbf{x}$ represents the grid spacing. A turbulent field is numerically forced by injecting energy into the low wave-number Fourier modes (e.g., Sigia and Patterson [31] and Molßen et al. [32]). For more details about the initialization of the flow field and the forcing method used in this study, see Abdel Kareem et al. [30].

The dissipation length scale is in an isotropic turbulent flow are the integral length scale, the Taylor micro-scale, and the Kolmogorov micro-scale. The integral length scale $\ell$ characterizes the energy-containing scales, whose definition is $\ell = \frac{2}{c} \int \frac{f_{eq}^s \mathbf{u} \cdot \mathbf{x}}{\rho \mathbf{x}} \, dk$, where the energy spectrum function $E(k)$ at the scalar wave number $k \equiv \langle \mathbf{k} \cdot \mathbf{k} \rangle^{1/2}$, whose definition in spectral space is $E(k) = \frac{1}{2} \sum_{\mathbf{k}, \mathbf{k}'} |\mathbf{k} \cdot \mathbf{k}'|^{1/2} \langle E(k) \rangle$, and $\mathbf{u}$ are the Fourier coefficients of the velocity. The Kolmogorov micro-scale $\eta$ is defined as $\eta = \left( \frac{\nu}{\epsilon} \right)^{1/4}$, where $\epsilon$ is the average energy dissipation rate. Finally, the Taylor micro-scale is $\lambda = \sqrt{\frac{16 \nu}{\epsilon}}$. The energy dissipation rate is defined as follows $\epsilon = 2 c S_t S_p$.

3. Curvelets

The curvelet transform was developed by Candès and Donoho [33]. The curvelet transform represents edges and singularities along a curve more precisely with needle-shaped basis elements with super directional sensitivity and smooth contour-capturing efficiency [34]. In this paper, we apply second generation curvelet
transform in 3D [35–36] for filtering a homogeneous isotropic turbulent flow.

3.1. Continuous curvelet transform

The continuous curvelet transform can be defined by the pair of windows $W(r)$ (a radial window) and $V(r)$ (an angular window), with the variable $W$ as a frequency domain variable and $r$ and $\theta$ as polar coordinates in the frequency domain, where

$$\sum_{j=-\infty}^{\infty} \mathbb{E}^2(2^{-j}r) = 1, \quad r > 0,$$  \hspace{1cm} (3) 

$$\sum_{t=-\infty}^{\infty} \mathbb{V}^2(t - 2\pi t) = 1, \quad t \in \mathbb{R}.$$  \hspace{1cm} (4)

A polar 'wedge' represented by $U_j$ is supported by $W$ and $V$, the radial and angular windows. $U_j$ is defined in the Fourier domain by

$$U_j(\omega) = W(2^{-j}r)V(2)(\theta - \theta_j),$$  \hspace{1cm} (5)

where

$$\theta_j = (2\pi t_j)2^j.$$  \hspace{1cm} (6)

The curvelet transform can be defined as a function of $x = (x_1, x_2)$ at scale $2^{-j}$, orientation $\theta_j$ and position $\mathbf{x}^{ij}(k)$

$$\phi_{j,k} = \phi_1(k - \mathbf{x}^{ij}(k)).$$  \hspace{1cm} (7)

$R_e$ denotes the rotation matrix with the angle $\theta$. The curvelet coefficients of the function $f \in L^2(\mathbb{R}^2)$ are simply the inner product between $f$ and $\phi_{j,k}$:

$$c_{j,k} = \langle \phi_{j,k}, f \rangle = \int_{\mathbb{R}^2} \bar{\phi}_{j,k}(x) f(x) dx.$$  \hspace{1cm} (8)

3.2. Discrete curvelet transform

The 3D discrete curvelet transform [36] takes as input a 3D Cartesian grid of the form $f(n_1, n_2, n_3)$, $0 < n_1, n_2, n_3 < n$ and outputs a collection of coefficients $c_{j,k}^\mathbf{b}$ defined by

$$c_{j,k}^\mathbf{b} = \sum_{n_1, n_2, n_3} f(n_1, n_2, n_3) \phi_{j,k}(n_1, n_2, n_3),$$  \hspace{1cm} (9)

where $j, \ell \in Z$ and $k = (k_1, k_2, k_3)$. Using a Cartesian corona of the form

$$\mathbf{W}_j = \phi_{j}(\mathbf{x}),$$  \hspace{1cm} (10)

and

$$W_j(\omega) = \sqrt{\phi_{j,1}^2(\omega) - \phi_{j,0}^2(\omega)}, \quad j \geq j_0,$$  \hspace{1cm} (11)

with

$$\phi_0(\mathbf{x}) = (\phi_0(\mathbf{x}_1), \phi_0(\mathbf{x}_2), \phi_0(\mathbf{x}_3)) = (\phi_0(\mathbf{x}_1) \phi_0(\mathbf{x}_2) \phi_0(\mathbf{x}_3)).$$  \hspace{1cm} (12)

The frequency window $\hat{U}_{j,0}$ is again defined by $\hat{U}_{j,0} = \hat{W}_j$. Suppose $\hat{U}_{j,0}$ is supported in a rectangular box of integer size $L_{1,k} \times L_{2,k} \times L_{3,k}$. The discrete curvelets at the coarsest level are defined by their Fourier transforms

$$\phi_{j,0,k}(\omega) = \hat{U}_{j,0}(\omega) \sqrt{L_{1,k} L_{2,k} L_{3,k}} e^{-2\pi i \left( \frac{n_1 k_x}{L_{1,k}} + \frac{n_2 k_y}{L_{2,k}} + \frac{n_3 k_z}{L_{3,k}} \right)},$$  \hspace{1cm} (13)

where $0 \leq k_1 < L_{1,k}$, $0 \leq k_2 < L_{2,k}$ and $0 \leq k_3 < L_{3,k}$. Every Cartesian corona has six components, one for each cube of the unit cube. Each component is regularly partitioned into $2^2 \cdot 2^2$ wedges with the same volume. Take the first component (corresponding to $W_1 > 0$) as an example. Suppose that for the $\ell$-th wedge, $(1, x, y)$ is the direction of the center line of the wedge. We define its smooth angular window as

$$\bar{V}_{k_1} = \bar{V} \left( 2^{1/2} \omega_2 \omega_3 \omega_1 \bar{V} \left( 2^{1/2} \omega_2 \omega_3 \omega_1 \right) \right).$$  \hspace{1cm} (14)

4. The 3D total variation filtering method

The filtering method proposed in this section is based on the Rudin–Osher–Fatemi (ROF) [37] total variation (TV) filtering technique. The problem of data noise removal or filtering is, given noisy data $f : \Omega \rightarrow \mathbb{R}$, to estimate the clean data $u$ for additive Gaussian noise, where the degradation model describing the relations between $f$ and $u$ is $f = u + \eta$.

4.1. The 3D Bregman TV filtering method

A general model for TV-regularized filtering is to find a $u$ that minimizes

$$\min_u \left( \int_{\Omega \cap B(V)} |\nabla u| dx + \frac{\lambda}{2} \int_{U} (u - f)^2 dx \right).$$  \hspace{1cm} (15)

Here, $\Omega$ and $D$ are bounded subsets of $\mathbb{R}^3$. $BV(\Omega)$ is the space of all bounded variations, and $\lambda$ is a positive parameter. This general model can be used to perform data filtering. Methods for efficiently solving TV regularized minimizations are a topic of ongoing research. One of the recent efficient methods is the split Bregman method of Goldstein and Osher [38]. In this paper, we present a method for filtering 3D volume data and apply it to turbulent flow. The split Bregman method solves the minimization problem by operator splitting followed by iteration to solve the split problem. The split problem is

$$\min_{u, d} \left( \int_{\Omega \cap B(V)} |\nabla u| dx + \frac{\lambda}{2} \int_{U} (u - f)^2 dx \right),$$  \hspace{1cm} (16)

where $d = \nabla u$. Bregman iteration is used to solve the split problem. In each iteration, Bregman iteration calls for a solution of the following problem

$$\min_{u, d} \left( \int_{\Omega \cap B(V)} |\nabla u| dx + \frac{\lambda}{2} \int_{U} (u - f)^2 dx + \frac{\mu}{2} \| \nabla u - d \|^2 \right),$$  \hspace{1cm} (17)

where the additional terms are quadratic penalties enforcing the constraints, and $\beta$ and $\beta$ are variables related to the Bregman iteration algorithm. Here, $\mu$ is an increasing positive constant representing the penalty function weight. We begin by addressing the anisotropic problem

$$\min_u \left( |\nabla^2 u | + |\nabla^2 u| + |\nabla^2 u| + \frac{\lambda}{2} |u - f|^2 \right)^2.$$  \hspace{1cm} (18)

To apply Bregman splitting, we put $d_1 = |\nabla^2 u|$, $d_2 = |\nabla^2 u|$ and $d_3 = |\nabla^2 u|$. Also for simplicity we put $|d_1| = |d_1| + |d_3| + |d_2|$. To weakly enforce the constraints in the formulation, we add a penalty function term (for details see [39]).

$$\min_{u, f, d_1, d_2, d_3} \left( |d_1| + \frac{\lambda}{2} |u - f|^2 + \frac{\mu}{2} \left( |d_1| + |\nabla^2 u| + |d_2| + |\nabla^2 u| + |d_3| + |\nabla^2 u| \right) \right).$$  \hspace{1cm} (19)

Let $X_d = d_1 - \nabla^2 u - b^1, Y_d = d_2 - \nabla^2 u - b^2$ and $Z_d = d_3 - \nabla^2 u - b^3$. Finally, we strictly enforce the constraints by applying the Bregman iteration to get

$$\min_{d_1, d_2, d_3} \left( |d_1| + \frac{\lambda}{2} |u - f|^2 + \frac{\mu}{2} \left( ||X_d||^2 + ||Y_d||^2 + ||Z_d||^2 \right) \right),$$  \hspace{1cm} (20)

where the positive values $b^1$, $b^2$ and $b^3$ are chosen through Bregman iteration. To solve this minimization problem, we will apply the
iteration minimization approach, which requires us to solve the
sub-problem
\[ u^{k+1} = \min_u \left( \frac{1}{2} \| f - f \|^2 + \frac{\mu}{2} \left( \| \nabla u \|^2 + \| \nabla \| \|^2 + \| \nabla z \|^2 \right) \right), \]  
(21)
with the optimality condition
\[ \nabla u^{k+1} - u \triangleq u^{k+1} = f + \mu \nabla^2 u (d_k^u - b_k^u) + \mu \nabla^2 u (d_k^u - b_k^u) \]
(22)

Then the solution can then be written as
\[ u_{ij}^{k+1} = c_{ij}^{k+1}, \]  
(23)
where
\[ c_{ij}^{k+1} = \frac{\mu}{\lambda + 6 \mu} (f_{ij} + u_{ik} + d_{ik} + b_{ik}), \]
(24)

\[ u_{ijk} = u_{i-1,j-1} + u_{i,j} + 1, d_{ij} = u_{i+1,j} + u_{i,j-1} + u_{i-1,j} + u_{i,j+1} - d_{ik} = d_{ik} - d_{ij}, \]
and \[ b_{ijk} = b_{ijk} - b_{ijk} - b_{ijk} \]  

The isotropic TV model can also be minimized using the split Bregman method. In this case, we wish to solve
\[ \min_u \left( \sum_i \sqrt{(\nabla_i u)^2 + (\nabla_j u)^2 + \frac{\mu}{2} \| f - f \|^2} \right). \]  
(25)

Just as we did for the anisotropic problem, we will split the components of the problem by setting \[ d_{ij} = |\nabla_i u| \] and \[ d_{ij} = |\nabla_j u| \]. The split Bregman formulation of the problem then becomes
\[ \min_{d_i, d_j} \left( \| d_i, d_j, d_k \|^2 + \frac{\mu}{2} \| f - f \|^2 + \frac{\mu}{2} \left( \| \nabla d_i \|^2 + \| \nabla d_j \|^2 + \| \nabla d_k \|^2 \right) \right). \]  
(26)

In order to apply the iterative minimization procedure to this problem we can still explicitly solve the minimization problem for \[ d^{k+1}, d^{k+1}, d^{k+1} \] using a generalized shrinkage formula
\[ d_{ik}^{k+1} = \max \left( \frac{\mu}{\lambda} - 1, 0 \right) \nabla u^{k} + b_k^u \]
(28)
\[ d_{ij}^{k+1} = \max \left( \frac{\mu}{\lambda} - 1, 0 \right) \nabla u^{k} + b_k^u \]
(29)
and
\[ d_{ij}^{k+1} = \max \left( \frac{\mu}{\lambda} - 1, 0 \right) \nabla u^{k} + b_k^u \]
(30)

where \[ s_k = \sqrt{|\nabla_i u|^2 + \| b_k^u \|^2 + \| b_k^u \|^2} \].

5. Results and discussion

The filtering methods are applied against two datasets of different grid sizes to examine their efficiencies and accuracies. Using low resolution results as a reference to analyze higher resolution results may give an insight to understand the universal features or the universality of coherent and incoherent statistics of homogeneous isotropic turbulence. Also, the difference between the results can give some idea on how to modify the parameters in the filtering method when the grid size is changed. The statistical results of the raw turbulent fields and the corresponding filtering fields can be organized in the following categories.

5.1. Characteristics of the turbulent fields

The Taylor–Reynolds number is one of the most important features that characterizes simulations of forced isotropic turbulence. In this paper the Taylor–Reynolds number \[ R_t = \frac{\bar{u} \bar{\lambda}}{\nu} \]. The statistical properties such as the Taylor micro-scale, the Kolmogorov micro-scale, and the integral length scale for the two cases are summarized in Table 1.

Table 1 Characteristics of turbulent fields.

<table>
<thead>
<tr>
<th></th>
<th>LBM-03Q19(128³)</th>
<th>LBM-03Q15(256³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_t )</td>
<td>72</td>
<td>180</td>
</tr>
<tr>
<td>( \ell )</td>
<td>1.67</td>
<td>1.0058</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.2751</td>
<td>0.2539</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0.017</td>
<td>0.0096</td>
</tr>
<tr>
<td>( k_{max} )</td>
<td>1.108</td>
<td>1.2288</td>
</tr>
</tbody>
</table>

It was reported based on experimental data that isotropic turbulence is obtained when spectra begin to collapse at high wave numbers under Kolmogorov scaling and when the Kolmogorov power law [41] is observed in the inertial sub-range. The short range of this scaling can be observed in Fig. 1, which depicts the energy spectrum of the chosen time instant reaching the steady state. More characteristics of the flow field and a comparison with the spectral method can be found in [30].

It is important to study the characterization and extraction of vortexal structures in order to understand the dynamics of turbulent flow fields. It has been reported numerically that an isotropic turbulent field is filled by tube-like vortices (e.g., Dubief and Declayere [42]; Lesieur and Ossia [43] and Miura and Kida [44]). Fig. 2 depicts the vortical structures observed using the high-rotational identification method (i.e., the Q-identification method) [45]. Here, Q is the second invariant of the velocity gradient tensor and is defined as

\[ Q = -\frac{1}{2} \left( S_{ij} S_{ij} - \Omega^2 Q_{ij} \right), \]  
(31)

where \[ S_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \] and \[ \Omega^2 = \frac{1}{2} (u_{ij} - u_{ji}) \] are the symmetric and the antisymmetric components of \( \nabla \). In this paper, Q is normalized by the average enstrophy of the flow field \( (Q_{\text{avg}}) \). For the higher resolution simulations (256³), the inertial sub-range becomes as the resolution increases for the Kolmogorov –5/3 power law at [41] as shown in Fig. 3, which depicts the energy spectrum of the chosen time instant reaching the steady state. The vortical structures are visualized in Fig. 4, where the turbulent field can be observed as groups of fine-scale structures.

5.2. Filtering results

The results of applying the two filtering methods, curvelet transform and total variation against the Q-field for the two simulations are organized as follows.

5.2.1. 3D curvelet filtering results

Curvelets are multi-dimensional wavelets that provide a new architecture for multiscale analysis. Unlike wavelets, curvelets are localized not only in the position and frequency domain but also in orientation. This property has caused the curvelet transform to become a promising tool for filtering problems [46–48]. The 3D discrete curvelet algorithm can be organized as follows.
(1) Apply the three-dimensional FFT to obtain the Fourier samples $\tilde{f}(x_1, x_2, x_3)$.
(2) For each scale $j$ and angle $\theta$, form the product $U_{j,\theta} f(x_1, x_2, x_3)$.
(3) Wrap this product around the origin and obtain $W(U_{j,\theta} f(x_1, x_2, x_3))$.
(4) Apply a 3D inverse FFT to each $W(U_{j,\theta} f)$, hence collecting the discrete coefficients $C_{j,\theta}$.

The steps of the curvelet filtering method can be summarized as follows:

(1) Apply curvelet transform to noisy 3D volume data.

(2) Apply hard thresholding to curvelet coefficients: Letting $y$ be the noisy curvelet coefficients, we use the following hard thresholding rule for estimating the unknown curvelet coefficients \cite{49}

\[ \tilde{y} = \begin{cases} y, & |y| \geq T \\ 0, & |y| < T \end{cases} \]

where $T = k r_n \sqrt{2 \ln N}$ is the thresholding parameter and depends on the variance of the noise $\sigma_n = \frac{M}{\sqrt{4 \ln N}}$ and on the size of the data $N$ \cite{50}. The variance of the noise is found to be 0.1665 for the $128^3$ and 0.1167 for the $256^3$. Here $k = 4$ for the first scale ($j = 1$), $k = 3$ for $j > 1$, and $M$ is the median of the curvelet coefficients of the highest frequency in the first scale.

(3) Apply inverse curvelet transform to the result of step 2.
Visualizations of the coherent (incoherent) parts \((128^3)\) obtained by the curvelet transform are presented in Fig. 5 (Fig. 6). The probability distribution function (PDF) of the coherent and incoherent parts in semi-logarithmic coordinates is given in Fig. 7, while the \(Q\)-thresholds obtained by the preceding procedures are listed in Table 2. It is clear that the vortical structures in Fig. 5 resemble those found in Fig. 2 and that the incoherent part does not contain any vortical structures as seen in Fig. 6. The PDF of the coherent part is found to be non-Gaussian, but the incoherent part is tailed Gaussian (i.e. almost Gaussian but with weaker tails). The PDF of the curvelet coherent part almost coincides with that of the non-filtered part. The Q-spectra of the non-filtered, coherent and incoherent fields are presented in Fig. 8. The incoherent spectrum spreads over all wave numbers and increases at high wave numbers.

For the \(256^3\) case, Fig. 9 represents the coherent part and Fig. 10 represents the incoherent part. It is clear that, for higher resolution, the curvelet filtering method captures the vortical structures smoothly without loss of the geometric shape of the structures. Also, no coherent structures are observed in the incoherent part. Fig. 11 presents the PDF of the non-filtered, coherent, and incoherent fields in semi-logarithmic coordinates, with the \(Q\)-thresholds listed in Table 2. The coherent part behaves similarly to the non-filtered field, and the incoherent part has an exponential PDF with much weaker tails. The Q-spectra of both the non-filtered and coherent parts are similar to each other, especially at low wave numbers. Also, the incoherent spectrum exists at low and high wave numbers as seen in Fig. 8 for the \(128^3\) case and Fig. 12 for the \(256^3\) case.

5.2.2. 3D total variation filtering results

The total variation method depends on the split Bregman method used in many image filtering applications [38]. The steps of the split Bregman TV filtering algorithm in 3D are.

\[
\begin{align*}
\text{Initialize} & \quad u_0 = f, \quad d_0 = b_0 = 0, \\
\text{While} & \quad \frac{\|u^k - u^{k-1}\|_2}{\|u^k\|_2} \geq \epsilon, \text{then} \\
& \quad u^{k+1} = \mathcal{Q}_{ijr}, \\
& \quad d_x^{k+1} = \max \left( \frac{g^k - 1}{\mu}, 0 \right) \nabla_x u^k + b_x^k, \\
& \quad d_y^{k+1} = \max \left( \frac{g^k - 1}{\mu}, 0 \right) \nabla_y u^k + b_y^k, \\
& \quad d_z^{k+1} = \max \left( \frac{g^k - 1}{\mu}, 0 \right) \nabla_z u^k + b_z^k, \\
& \quad b_x^{k+1} = b_x^k + (\nabla_x u^{k+1} - d_x^{k+1}), \\
& \quad b_y^{k+1} = b_y^k + (\nabla_y u^{k+1} - d_y^{k+1}), \\
& \quad b_z^{k+1} = b_z^k + (\nabla_z u^{k+1} - d_z^{k+1}), \\
\text{End.}
\end{align*}
\]

During the numerical steps, \(\lambda = 0.033\) and \(\mu = 3\lambda\) are used for the stability problem and convergence. The iterations converge at nine...
iterations where $\epsilon = 10^{-3}$. Visualizations of the coherent and incoherent parts ($128^3$) obtained by the total variation transform are presented in Figs. 13 and 14. The probability distribution function (PDF) of the coherent and incoherent parts is given in Fig. 15. The thresholds used for their extraction are listed in Table 3. It is clear that the vortical structures in Fig. 13 resemble those found in Fig. 2, and that the incoherent part does not contain any vortical structures as seen in Fig. 13. The PDF of the coherent part is found to be non-Gaussian, but in the incoherent part it is tailed Gaussian. The PDF of the coherent field almost coincides with that of the non-filtered part.

The TV results for the $256^3$ are given in Figs. 17–20. In this case, the TV method also isolates the coherent part (Fig. 17), with structures similar to the structures found in the non-filtered field.

### Table 2
Statistical properties for the $128^3$ with variance $\sigma_x = 0.1665$.

<table>
<thead>
<tr>
<th></th>
<th>$Q_T$</th>
<th>$Q_C$-curvelet</th>
<th>$Q_C$-TV</th>
<th>$Q_I$-TV</th>
</tr>
</thead>
<tbody>
<tr>
<td>% Of coefficients</td>
<td>100</td>
<td>4.3</td>
<td>95.7</td>
<td>4.6</td>
</tr>
<tr>
<td>Skewness</td>
<td>−0.003</td>
<td>−0.01</td>
<td>−0.008</td>
<td>−0.013</td>
</tr>
<tr>
<td>Flatness</td>
<td>18.56</td>
<td>26.3</td>
<td>1.9</td>
<td>27.5</td>
</tr>
<tr>
<td>$Q_{max}$</td>
<td>150.82</td>
<td>146.07</td>
<td>6.51</td>
<td>145.2</td>
</tr>
<tr>
<td>$Q_{min}$</td>
<td>−49.91</td>
<td>−45.79</td>
<td>−5.2</td>
<td>−45.1</td>
</tr>
<tr>
<td>Threshold</td>
<td>4.5</td>
<td>4.5</td>
<td>2.5</td>
<td>4.5</td>
</tr>
</tbody>
</table>

Fig. 8. Spectra of the nonfiltered, coherent and incoherent fields (curvelet decomposition: $128^3$).

Fig. 9. Isosurfaces of the coherent field (curvelet decomposition: $256^3$): $\frac{\sigma_x}{\sigma_T} = 5.98$.

Fig. 10. Isosurfaces of the incoherent field (curvelet decomposition: $256^3$): $\frac{\sigma_x}{\sigma_I} = 1.5$.

Fig. 11. PDF of the nonfiltered, coherent and incoherent fields (curvelet decomposition: $256^3$).
Clearly, no vortical structures are found in the incoherent part (Fig. 18). The PDF in this case is presented in Fig. 19, with the \( Q \) thresholds listed in Table 3. The coherent PDF is found to be similar to the PDF of the non-filtered field. The incoherent PDF is almost Gaussian but with weaker tails. The \( Q \)-spectra of the coherent and incoherent fields are compared with the non-filtered field \( Q \)-spectrum in Fig. 20. The figure demonstrates that the coherent spectrum is similar to the non-filtered field spectrum. The incoherent field in the TV case also exists at low and high wave numbers, which is similar to the curvelet filtering results.

### 5.3. Comparison between the two filtering methods

Previous efforts to extract the incoherent part were introduced by Farge et al. [51]. Their research identified the coherent and incoherent fields using wavelet filtering and then compared their results to the results of Fourier filtering. They found that wavelet filtering captures the important features of the coherent structures but that Fourier filtering identifies coherent structures in both fields, so they refer to these as large- and small-scale. Although their wavelet and Fourier decompositions were applied to the vorticity field, the definition and statistical behavior of the incoherent part was almost the same. However, the present two filtering methods preserve the edges and structures better than wavelet-based methods. It should be pointed out that we cannot find a major difference between the results obtained by the two proposed filtering methods. They extract almost identical incoherent flows, and the coherent structures are almost identical to the non-filtered field. Also, in the curvelet and TV 256\(^3\) cases, the PDF curves of the coherent and the non-filtered fields are almost the same, however there is a slight difference between the two curves in the 128\(^3\) case.
In the 128³ case, it can be concluded that the Q-spectra of the coherent and non-filtered fields are almost the same at low wave numbers (approximately $k\eta = 0.13(k \lesssim 8)$ in the curvelet case and $k\eta = 0.1(k \lesssim 6)$ in the total variation case) and that they then diverge from each other. At the same time, the incoherent spectra starts to build at the same wave numbers, specifically $k = 8.0$ and $k = 6.0$ in the curvelet and total variation methods, respectively. In the 256³ case, the two filtering methods exhibit almost identical results, with the Q-spectra of the coherent fields and the non-filtered field diverging from each other at $k\eta = 0.33$ and $k\eta = 0.32$ for curvelet and TV, respectively. The following tables summarize some statistical properties of the extracted fields (non-filtered, Q; coherent, QC; and incoherent, QI) using the two filtering methods for the 128³ and 256³ cases. The curvelet coefficients for the coherent and incoherent fields as well as the coefficient modes of the TV, the skewness and the flatness, are also given.

Table 3
Statistical properties for the 256³ with variance $\sigma_r^2 = 0.1167$.

<table>
<thead>
<tr>
<th></th>
<th>$Q_r$</th>
<th>$Q_r$-curvelet</th>
<th>$Q_r$-TV</th>
<th>$Q_c$-TV</th>
</tr>
</thead>
<tbody>
<tr>
<td>% Of coefficients</td>
<td>100</td>
<td>3.5</td>
<td>96.5</td>
<td>3.9</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.25</td>
<td>0.25</td>
<td>0.000</td>
<td>0.25</td>
</tr>
<tr>
<td>Flatness</td>
<td>47.1</td>
<td>45.5</td>
<td>1.65</td>
<td>47.1</td>
</tr>
<tr>
<td>$Q_{\max}$</td>
<td>174.308</td>
<td>174.5</td>
<td>3.62</td>
<td>174.1</td>
</tr>
<tr>
<td>$Q_{\min}$</td>
<td>-48.24</td>
<td>-48.78</td>
<td>-3.0</td>
<td>-48.43</td>
</tr>
<tr>
<td>Threshold</td>
<td>5.98</td>
<td>5.98</td>
<td>1.5</td>
<td>5.98</td>
</tr>
</tbody>
</table>

In the 128³ case, it can be concluded that the Q-spectra of the coherent and non-filtered fields are almost the same at low wave numbers (approximately $k\eta = 0.13(k \lesssim 8)$ in the curvelet case and $k\eta = 0.1(k \lesssim 6)$ in the total variation case) and that they then diverge from each other. At the same time, the incoherent spectra starts to build at the same wave numbers, specifically $k = 8.0$ and $k = 6.0$ in the curvelet and total variation methods, respectively. In the 256³ case, the two filtering methods exhibit almost identical results, with the Q-spectra of the coherent fields and the non-filtered field diverging from each other at $k\eta = 0.33$ and $k\eta = 0.32$ for curvelet and TV, respectively. The following tables summarize some statistical properties of the extracted fields (non-filtered, Q; coherent, QC; and incoherent, QI) using the two filtering methods for the 128³ and 256³ cases. The curvelet coefficients for the coherent and incoherent fields as well as the coefficient modes of the TV, the skewness and the flatness, are also given.

The skewness of the non-filtered field in the case of 128³ is about zero, and both curvelet and TV preserve this property (Table 2). The
The Lattice Boltzmann method with resolutions of $128^3$ and $256^3$ implemented in 3D. The turbulent flow fields were simulated using different filtering methods such as wavelet and partial differential equation filtering methods. These are the goals of future research which can be applied to higher resolution DNS of turbulent flows.

6. Conclusion

The curvelet and total variation filtering methods were used to decompose two forced turbulent flow fields into coherent and incoherent parts. The total variation Bregman method was numerically implemented in 3D. The turbulent flow fields were simulated using the Lattice Boltzmann method with resolutions of $128^3$ and $256^3$. The vortical structures are extracted using the high-rotational method and many fine tubes are visualized. The two filtering methods are efficient tools for extracting coherent vortices out of turbulent flows. In both filtering cases, the coherent vortical structures are identified without any loss of the physical structures in the turbulent fields and are found to be identical to the total fields. The spectra of the coherent and incoherent fields show that the incoherent spectra start to build at the same wave number where the spectra of the coherent and incoherent fields show that the incoherent fields and are found to be identical to the total fields. The curvelet transform for image denoising. IEEE Trans Image Process 2000;11:670–84.

Skewness in the case of $256^3$ resolution (Table 3) is also close to zero for the incoherent part.

References


