Some Calculations on Link Polynomials from 2-Parameter Quantum Groups

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Abstract—Starting with the link invariant \( G(p,q,z) \), a skein relation is introduced which enables us to calculate the polynomial starting with \( G(p,q,z) = 1 \), for the unknot. The relation between \( G(p,q,z) \) and the known 2-variable link invariant \( K(l,m) \) is shown. Then the polynomial of some common knots is calculated. Also it is shown that this invariant failed to distinguish the Birman’s pairs as well as Jones invariant.

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1. INTRODUCTION

A braid on \( n \) strings is an embedding of \( n \) oriented intervals (strings) in \( D^2 \times I \) joining \( n \) points in the disc \( D^2 \times \{0\} \) to the \( n \) points in \( D^2 \times \{1\} \). The strings may be permuted but the last coordinate increases monotonically on each string Fig. 1a. The set of all such braids form a group, Artin braid group \( B_n \), which is presented by generators \( \sigma_i \), Fig. 1b, \( i = 1,2,\ldots,n-1 \) and relations

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| \geq 2
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}
\] (2)

The closure \( \tilde{x} \) of a braid \( x \) formed by joining the top points to the bottom as shown in Fig. 1c, and the braid \( x \in B_n \) is called the \( n \) braid representative of the knot (or link) \( L \approx \tilde{x} \).

The closure of braids which are equivalent under a sequence of moves of the following types are isotopic links,

\[
(x,n) \rightarrow (\beta x\beta^{-1},n), \quad \text{for any } \beta \in B_n
\]

\[
(x,n) \rightarrow (x\sigma_n^{\pm 1},n+1)
\] (4)

Fig. 1. (a) Geometric braid. (b) Positive and negative crossings. (c) Closed braid.
Two knots $K_1$ and $K_2$ in $S^3(R)$ belong to the same isotopy type if there exists an orientation preserving homeomorphism of $S^3$ which maps $K_1$ onto $K_2$.

Recently El-Rifai, Hegazi and Ahmed introduced a new polynomial invariant $G_I(p,q,z)$ which is an invariant of the isotopy type of an oriented knot and link $L \subseteq S^3$ [1]. This polynomial invariant is based on some algebraic structures of the 2-parameter quantum group, which as a symmetry group of the 2-parameter quantum plane, where the 2-parameter deformation $qAB - pBA = C$ is in correspondence with the 2-parameter quantum group [2, 3].

For a link $L$, the polynomial invariant $G_I(p,q,z)$ is given by,

$$G_L(p,q,z) = h^{e-n+1}z^{-e} \text{tr}(\pi(x))$$

(5)

where $\pi: B_n \rightarrow H(p,q,n)$, $\pi(e_i) = q^{e_i}i$ is a representation of the braid group $B_n$ in the Hecke algebra $H(p,q,n)$ generated by $g_i = P^{-1} R_i^{-1}$, $i = 1,2,\ldots,n-1$ and the associated relations:

$$g_ig_j = g_jg_i \quad |i - j| \geq 2$$

$$g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$$

$$g_i^2 = (\frac{1}{pq} - 1)g_i + \frac{1}{pq}$$

where $p,q$ are free parameters and $R$ is a universal $4 \times 4$ $R$-matrix, which can be obtained by solving the Yang Baxter equation [3] and

$$\text{Tr}: H(p,q,n) \rightarrow C$$

(6)

For every complex number, $c \in \mathbb{C}$, there is a trace function [4], uniquely defined by linearity and

$$\text{tr}(I) = 1$$

$$\text{tr}(ab) = \text{tr}(ba)$$

$$\text{tr}(wg_{a+1}) = z \text{tr}(w), w \in H(p,q,n)$$

and $z$ is a closed $n$-braid representative of $L$ in $B_n$. Finally the number $e$ is the exponent sum of the braid $a$, and $h = z[pq(z+1) - 1]$. For details in braid groups and knots we refer to [5]. An excellent work for quantum groups of one parameter, and knot invariants was done by L. Kauffman [6].

Our structure of this paper is as follows. We do some calculations on the knot polynomial $G(p,q,z)$; by introducing the skin relation which enables us to compute $G(p,q,z)$. And then give the relationship between this polynomial and other known polynomials. Finally, we computed the polynomial for the trefoil and figure eight knots. Also we show that the given invariant does not distinguish the Birman’s pairs.

Suppose that $L_+, L_-$ and $L_0$ are regular projections of three oriented links that are exactly the same except near one point as in Fig. 2:

Let $O$ denote the unknot of one component then, from eqn (5), $G_o(p,q,z) = 1$. Then the skel relation induction process associated to the polynomial $G(p,q,z)$, provides an alternative method for calculating the polynomial.

**Theorem:** The polynomial $G(p,q,z)$ satisfies the formula

$$z^2pqG_{L_+} - h^2G_{L_-} = (1-pq)zhG_{L_0}$$

(7)
Proof:

Let $L_+$ have a braid representative $x \sigma_1^2$ in $B_n$, then $x$ and $x \sigma_i$ are braid representatives of $L_-$ and $L_0$, respectively.

But

$$g_1^2 = \left(\frac{1}{pq} - 1\right)g_1 + \frac{1}{pq}$$

Then

$$\pi(x)g_1^2 = \left(\frac{1}{pq} - 1\right)\pi(x)g_1 + \pi(x)\frac{1}{pq}$$

Then

$$tr(\pi(x)g_1^2) - \frac{1}{pq} tr(\pi(x)) = \left(\frac{1}{pq} - 1\right)tr(\pi(x)g_1)$$

Multiplying by $h^{r-n-1} \xi^{-r}$ where $e$ is the exponent sum of $x$, then

$$h^{r-n-1} \xi^{-r} tr(\pi(x)g_1^2) - h^{r-n-1} \xi^{-r} \frac{1}{pq} tr(\pi(x)) = \left(\frac{1}{pq} - 1\right)h^{r-n-1} \xi^{-r} tr(\pi(x)g_1)$$

Hence eqn (5) implies that

$$\left(\frac{z}{\hbar}\right)^2 G_{L_+} - \frac{1}{pq} G_{L_-} = \left(\frac{1}{pq} - 1\right)\left(\frac{z}{\hbar}\right)G_{L_0}$$

Jones introduced a knot invariant $V_L(t)$ by using representations of the braid groups [7], then Ocneanu, Lickorish, Millett, Freyd, Yetter and Hoste [8], showed that $V_L(t)$ is a specialization of a 2-variable invariant $K_L(t,m)$, which it also specializes to the Conway polynomial $V_L(z)$, hence the Alexander polynomial $\Delta_L(t)$. The relationship between such invariants is given in [9] by Lickorish. The next result gives the relationship between $G(p,q,z)$ and the polynomial $K_L(t,m)$, hence to the other knot polynomials. The idea is to follow the algebras which lead to these polynomials.

Lemma:

$$K_L\left(\frac{ih}{z\sqrt{pq}}, i\left(\frac{1}{\sqrt{pq}} - \sqrt{pq}\right)\right) = G_L(p,q,z)$$
2. CALCULATIONS

1. The n-component unlink $L$, of unknots which has the identity $I$, as a closed $n$-braid representative then, $G_L(p,q,z) = h^{-n+1}z^n$. $Tr(1) = h^{-1}$

2. The right-handed trefoil, which has $\varepsilon = \sigma_1^3$ as a closed 2-braid representative, then $e = 3$, $n = 2$, hence

$$G_L(p,q,z) = h^2z^{-3} tr(g_1^3) = h^2z^{-3} \left[ \left( \frac{1}{p^2q^2} - \frac{1}{pq} + 1 \right)z + \left( \frac{1}{p^2q^2} - \frac{1}{pq} \right) \right]$$

3. The figure eight knot, which has $\varepsilon = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ as a closed 3-braid representative, then $e = 0$, $n = 3$, hence $G_L(p,q,z) = h^{-2}Tr(g_1^2g_2^{-1})^2$

$$= h^{-2} \left[ (-p^2q^2 + 3pq - 1)z^2 + \left( -p^2q^2 + 4pq - 4 + \frac{1}{pq} \right)z + \left( pq - 2 + \frac{1}{pq} \right) \right]$$

4. Birman [10] gave a general example of different knot types, having the same Jones polynomial, such as $\varepsilon = \Delta^{-2}\varepsilon$ and $\delta$ such that $e(\delta) = 6r, r \neq 0$ and $\delta = \sigma_1^r \sigma_2^r \sigma_1^r \sigma_2^r \ldots \sigma_1^r \sigma_2^r p_r^r$. $p_r^r$ $\rightarrow$ $q_r^r$. Now, let $\varepsilon = A_2^r \sigma_2^{-1} \sigma_2$ and $\delta = \sigma_1^r \sigma_2^{-1}$, then $\hat{\varepsilon}, \hat{\delta}$ represent two different knot types, simply because they have different braid index, i.e. $\hat{\varepsilon}$ can not be represented as a 2-closed braid, but $\hat{\delta}$ can be represented as a 2-closed braid. But, according to Birman, $\hat{\varepsilon}$ and $\hat{\delta}$ have the same Jones polynomial. Unfortunately they have the same knot invariant $G(p,q,z)$, which is

$$G(p,q,z) = h^3 (pqz)^{-6} \left( (z+1)[1-pq+p^2q^2-p^3q^2+p^4q^4-p^5q^4] + z[p^6q^6] \right).$$

H. Kauffman had suggested a many variable link invariant, which is a master polynomial for all other link polynomials. We hope the work of J. Hodges and others for multiparameter quantum groups [11] is way to do that.

REFERENCES