Maximal Characterization and Series Function of Hardy-Sobolev spaces with an Application on Manifolds

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Abstract: Let M be a complete, non-compact Riemannian manifold, provided with a doubling measurableµ. In this paper we compared the maximal Hardy-Sobolev spaces with the Hajłasz Sobolev space on M, and we showed that they can be identified under the assumption of a Poincare inequality. The proof was based on a characterization of L_p on metric-measure spaces.

Key words: Hardy-Sobolev space, Hajłasz- Sobolev space, metric measure spaces.

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I. Introduction and Preliminaries

The series function in the Hardy-Sobolev space if its derivatives lie in the real Hardy spaceL_1, means that a maximal series function of the derivatives is integrated. One of the aims of this paper is, how to define the maximal series function of the derivatives off_r.

For a locally integrated series function f_r on M define the gradient in the sense of distributions, implies

$$\sum_{r=1}^{\infty} (\nabla f_r), \varphi := - \sum_{r=1}^{\infty} \int f_r \, \text{div} \, \varphi \, df \, \mu$$

(1)

For all smooth vector fields \varphi of compact support. Here \text{div} \varphi is the divergence, defined via z acting on 1-forms. Following the ideas from the scalar case see [4,17], a natural grand maximal a series functions would be to take, at a point x_1 \in M,

$$\sup \left| \sum_{r=1}^{\infty} f_r \, \text{div} \, \varphi \, df \, \mu \right|$$

where the supermom is taken over some family \mathcal{T}_1(x_1) of test vector fields \varphi. In [3] defined in terms of atomic decomposition, with an L_1-Sobolev space defined by Hajłasz (H^1) [8], we identified df for f_r \in H^1 with elements of the particulate L (Hardy space) of differential forms defined in [1] and use the usual maximal function characterization of H^1 (see [4, 15, 12]).

Out of order to do this; we need to extend the notion of divergence to a broader class of test vector fields. Here we defined a maximal a series functions(\nabla f_r)^+, where the test vector fields were, in a sense, only Lipschitz continuous. Furthermore, it was explained that for f_r \in L^{1}_{max}(M), (\nabla f_r)^+ \leq N f_r, at every point of M, and therefore a series function f_r in the homogeneous Hajłasz Sobolev space H^1_{1-r}, (see [11]) characterized by the condition Nf_r \in L, also satisfies(\nabla f_r)^+ \in L.

There is difficulty getting the converse, namely, introduce that a series function f_r with (\nabla f_r)^+ \in L_1_{1-r}(M) belongs H^1_{1-r}, either by controlling Nf_r or via an atomic decomposition. In appointed, when effort to do this, the trouble here of writing a given test a series functions η_r, with∫ η_r = 0, as the divergence of enough smooth vector field of compactness support. In the Euclidean setting, this can be done by a simple well-known construction including rephrased integration with respect to the coordinates (see [4, 5]) which preserves the smoothness with no gain. However, adapting such a construction to a manifold with constants which are independent of the local coordinates is not evident. In addition, if one wants to have a gain of derivatives, the case off_r = 0, which corresponds to starting with η_r \in L^1 and obtaining a vector field whose components have bounded derivatives, is not possible ([18]).

In Part 2, we define a new Hardy-Sobolev maximal a series functions (\nabla f_r)^+, which coincides with that (\nabla f_r)^+ used in [2] to define Hardy-Sobolev spaces on Lipschitz domains inR^n, and use it to define the homogeneous maximal Hardy-Sobolev space H^1_{1-r,max}. In Part 3, we compare this space with the homogeneous
Hajłasz Sobolev space $\mathcal{H}L_{p}$. We showed main result, Theorem (3.4), the proof of that, based on Proposition (4.1), is contained in part 4.

We work on completeness, non-compactness Riemannian manifold $M$. With the distance a series functions $\rho$, and the measure $\mu$ (volume) given by the Riemannian $\mathbb{R}^n$ structure, we view $(M, \rho, \mu)$ as a metric measure space, and use $B(x_i, s)$ to denote the metric ball of radius $s > 0$ centered at $x_i \in M$. Denote by $(\langle \cdot \rangle_{x_i})$ the Riemannian metric on the tangent space $T_{x_i}M$, let $T_{x_i}M$ be the cotangent space at $x_i$, and $d$ the exterior derivative. For a smooth a series functions $f_r$, the gradient $\nabla f_r$ can be viewed as the image of the 1-form $df_r$ under the isomorphism between $T^*_{x_i}M$ and $T_{x_i}M$ (see [4, 18]).

A series functions will be called Lipschitz continuous, denoted $f_r \in \text{Lip}(M)$, if there exists $C < \infty$ such that

$$\sum_{r} |f_r(x_i) - f_r(x_{(i-1)})| \leq \sum_{r} C \rho_r(x_i, x_{(i-1)}) \quad \forall x_i, x_{(i-1)} \in M.$$ 

The smallest such constant $C$ will be denoted by $\|f_r\|_{\text{Lip}}$. By $\text{Lip}_0(M)$ we will mean the space of compactly supported Lipschitz functions.

We will assume the measurable $\mu$ on $M$ satisfies the following.

**Definition 1.1.** Let $C > 0$, for $M$ be a Riemannian manifold, such that for all balls $B(x_i, s), x_i \in M, \sigma > 0$ we have

$$\mu(B(x_i, 2\sigma)) \leq C \mu(B(x_i, \sigma)).$$  \hspace{1cm} (2)

Notice that if $M$ satisfies (2) then

$$\dim(M) < \infty, \quad \mu(M) < \infty.$$ \hspace{1cm} (3)

**Lemma 1.2.** (see [3, 4]) Let $M$ be a Riemannian manifold satisfying (2), $\tau = \log_1 C_2, \theta \geq 1$. Then for all $(x_i, x_{(i-1)}) \in M$,

$$\mu(B(x_i, \theta R)) \leq C \theta^\tau \mu(B(x_i, R)).$$ 

We show definition concern to Poincare inequality on $M$.

**Definition 1.3.** (see [4]) Let $M$ a Riemannian manifold admits a Poincare inequality (2) for some $\varepsilon \geq 0$ if there exists a constant $C > 0$ such that, for every ball $B$ so $s > 0$.

$$\sum_{r} \left( \int_B |f_r - (f_r)_B|^{1-\varepsilon+\delta} \, d\mu \right)^{1/(1-\varepsilon+\delta)} \leq \sum_{r} C \sigma \left( \int_B |\nabla f_r|^{1-\varepsilon+\delta} \, d\mu \right)^{1/(1-\varepsilon+\delta)} \hspace{1cm} \text{(3)}$$

Whenever $f_r$ and its distributional gradient $\nabla f_r$ are $(1 - \varepsilon)$-integrated on $B$.

**II. New definition of maximal Hardy – Sobolev Space**

From [4], we define a new Hardy-Sobolev space maximal a series functions. Let us first recall the following definition.

**Definition 2.1.** (See [4]). Let $f_r \in L_{1-\varepsilon, \text{loc}}(M)$, we define its great maximal a series functions, that means by $(\nabla f_r)^+$ as pursued:

$$\sum_{r} (\nabla f_r)^+ (x_i) := \sup \left| \int_{M} f_r \varphi_r \, d\mu \right|,$$  \hspace{1cm} (4)

so $\varphi \in \text{Lip}_0(M)$ such that for some ball $B := B(x_i, s)$ includes backing $\varphi$.

$$\|\varphi\|_{\infty} \leq \frac{1}{\mu(B)}, \quad \|\nabla \varphi\|_{\infty} \leq \frac{1}{s \mu(B)},$$ \hspace{1cm} (5)

where

$$\|\varphi\|_{\infty} \leq 1.$$ 

Now we define the divergenced $\text{div} \psi \in C^\infty(M)$, by given a smooth vector field $\varphi$ with compactness support, so that

$$\int_{M} \sum_{r} (\nabla f_r, \varphi)_{x_i} \, d\mu = - \int_{M} \sum_{r} f_r \, \text{div} \psi \, d\mu,$$

and extend this to a locally integrated a series functions $f_r$ on $M$, in order to define $\nabla f_r$, in the sense of dividend, wherein (1). If this divisional slope coincides with a measurable vector-field valued a series functions, which we again denote by $\nabla(f_r)$, we can take its length in the Riemannian metric,$|\nabla f_r|_{x_i} := \langle \nabla(f_r) \rangle_{x_i}$, and compute the semi-norms,

$$\sum_{r} \|\nabla f_r\|_{1+\delta-\varepsilon} := \sum_{r} \left( \int_{M} |\nabla f_r|^{1+\delta-\varepsilon} \, d\mu \right)^{1/(1+\delta-\varepsilon)}, \quad \delta - \varepsilon \geq 0.$$

See quantity to $\varphi$ and $\psi$, so
\[
\sum (\nabla (f_r)^+ (x_i)) = \sup \left| \int \sum_r (\nabla \varphi, \psi)_{x_i} + \varphi \text{div} \psi) d\mu \right|,
\]
where \(B\) is the radius of the ball \(B\), we have
\[
\sup \varphi \in B, \quad \|\varphi\|_{\infty} \leq \frac{1}{\mu(B)}, \quad \|\nabla \varphi\|_{\infty} \leq \frac{1}{s\mu(B)}
\]
Observed to both \(\varphi, \psi\) are smooth, the quantity \(\left(\nabla \varphi, \psi, \varphi \text{div} \psi\right)\), idealizes the divergence of the product \(\varphi \psi\). See [4, 15, 12] consider fractional derivatives.

**Definition 2.2.** Let \(f_r \in L^\alpha(\chi), \chi\) denote to domain in \(M\), and \(\psi\) be a vector field in \(L^\alpha(\chi, TM)\) we say that in the distributional sense if there exists \(g_r \in L(\chi, M) \subset L_1(M)\), such that
\[
\int_M \sum_r f_r g_r d\mu = -\int_M \sum_r (\nabla f_r, g_r)_{x_i} d\mu,
\]
for all \(f_r \in L_{1-\varepsilon, loc} (M)\), with and its distributional gradient \(\nabla f\) integrable on \(\chi\).

**Remarks 2.3.**
(i) If \(M\) is a completeness non-compactness Riemannian manifold satisfying in (2) then \(\mu(M) = \infty\) and \(H_{1-\varepsilon} L \subset S^1\).
(ii) \(L_{\max}^\prime (M) = \{f_r \in L_{1-\varepsilon, loc} (M); (f_r)^+ \in L_1(M)\}\), when we used lebesgue theorem deduce \(L_{\max}^\prime (M) \subset L_1(M)\). The divergence \(\text{div} \psi \in C^\alpha(M)\) so that define,
\[
\sum_r \|f_r \text{div} \psi\|_{\infty} \leq \frac{1}{s}
\]
(See [4, 16]).

**Corollary 2.4.** Once \(M\) satisfies see Definition 1.3, \(\tau < 0\). Implies,
\[
\sum_r \left(\int_B |f_r(x) - f_r(x)|_{\gamma}^{1-\varepsilon} d\mu\right)^{1/\tau} \leq C\sigma \sum \left(\int_B \|\nabla f_r(x)\|_{1-\varepsilon} d\mu\right)^{1/\tau}, \quad \tau < 0.
\]

The maximal a series functions characterization of the Hardy-Sobolev space \(L_{1-\varepsilon}^\prime\), shown: a series functions in the Hardy-Sobolev space in order to Euclidean case if its derivatives lie in the real Hardy space \(L_1\), in the sense that a maximal a series functions of the derivatives is integral.

The homogeneous Hardy-Sobolev space \(HL_{1-\varepsilon}^\prime\) in the Euclidean case includes all locally integrated a series functions \(f_r\), such that \(\nabla (f_r) \in L(\|\|), \) some definitions can be displayed for this.

**Definition 2.5.** Let \(\phi\) be vector fields, \(\phi \in \eta(x_i)\) for some ball \(B, s(B)\) its radius
\[
HL_{1-\varepsilon, max}^\prime = \{f_r \in L_{1-\varepsilon, loc}; N(\nabla f_r) \in L_1\}.
\]
So \(HL_{1-\varepsilon, max}^\prime\) denote to maximal homogeneous Hardy-Sobolev space, where \(N(\nabla (f_r))\) is given by
\[
\sum_r N(\nabla (f_r)) = \sup \left| \int \sum_r (f_r)_{x_i} \text{div} \phi d\mu \right|.
\]
That is \(\phi \in L(\|\|\), \(TM\),
\[
\|\phi\|_{\infty} \leq \frac{1}{\mu(B)}, \quad \|N\|_{\infty} \leq \frac{1}{s\mu(B)}
\]
We equip this space with the semi-norm
\[
\sum_r \|f_r\|_{HL_{1-\varepsilon, max}} = \sum_r \|N(\nabla f_r(x))\|_{x_i}.
\]
Note that the definition of \(N(\nabla f_r)\) coincides with that of the maximal functions series \(N^\prime) f_r\) used in [8], to define Hardy-Sobolev spaces on Lipschitz domains in \(\mathbb{R}^\theta\).

We control the maximal a series functions \((\nabla f_r)^+ (x_i)\) and incline of \(f_r\) in the Point wise sense. Shown following:

**Proposition 2.6.** Let \(f_r \in L_{1-\varepsilon, max} (M)\) and \((\nabla f_r)^+ \in L_1(M)\) primarily defined by (1), is given by a series functions and gratifies,
\[
\sum_r |\nabla f_r|_{x_i} \leq C \sum_r (\nabla f_r)^+ (x_i) \quad \mu - a.e. x_i.
\]
Consequently,
\[ \tilde{H}L_{1-\varepsilon} \subset S^1_1, \]

with
\[ \sum_r \| (f_r)_x \|_{S^1_1} \leq C \sum_r \| \nabla (f_r)_x \|_{\tilde{H}L_{1-\varepsilon}}. \]

The non-homogeneous Sobolev space \( H^{1-\varepsilon}_L \) is defined as the space of \( f_r \) in \( L^p(M,\mu) \) with \( \sum_r \| \nabla (f_r)_x \|_{L_1} < \infty \). Similarly, we can define the homogeneous space \( \tilde{H} \) by taking only \( f_r \in L^{1-\varepsilon,loc}(M) \) with \( \| \nabla (f_r)_x \|_{L_1} < \infty \), and considering the resulting space modulo constants. Show define the new maximal homogeneous Sobolev space \( \check{H}L^{1-\varepsilon,\max}_L \).

**Definition 2.7.** Let \( Q \) is constant for supremum that is \( \psi \in L^m(B,M) \) to some \( B := B(Q,s) \), follows
\[ \check{H}L^{1-\varepsilon,\max} = \{ f_r \in L^{1-\varepsilon,loc}_L : M^+(\nabla f_r) \in L^1 \}, \]
where \( M^+(\nabla f_r) \) is given by
\[ \sum_r M^+(\nabla f_r)(x) := \sup_Q \int_Q \sum_r f_r \cdot \nabla \psi \, d\mu \bigg|_Q. \]

We equip this space with the semi-norm
\[ \sum_r \| (f_r)_x \|_{\check{H}L^{1-\varepsilon,\max}_L} = \sum_r \| M^+(\nabla f_r)_x \|_Q, \quad Q \leq 1. \]

In the introduction we have already noticed that \( M^+(\nabla f_r) \) coincides with that of the maximal series function \( M^{(1)}(f_r) \) used in \([2,4]\), to define Hardy-Sobolev spaces on Lipschitz domains in \( \mathbb{R}^n \).

### III. The maximal Hardy-Sobolev space comparison with Hajłasz Sobolev space

As in the homogeneous case, \( \tilde{H}L^{1-\varepsilon}_L \subset S^1_1 \), first we define that on metric measurable space \((X,d_{1-\varepsilon},m)\):

**Definition 3.1.** (Hajłasz). Let \( \varepsilon \geq 0 \). The homogeneous Sobolev space \( \tilde{H}L^{1-\varepsilon}_L \) is the set of all \( \epsilon \)-series functions \( u^\epsilon \in L^{1-\varepsilon,loc}_1 \) such that there exists a measurable \( \epsilon \)-series function \( \tau \geq 0, \tau \in L^{1-\varepsilon} \), satisfying
\[ \sum_r |u^\epsilon(x_i) - u^\epsilon(x_{i-1})| \leq d \sum_r \left( \tau(x_i) + \tau(x_{i-1}) \right), \quad \tau - a.e. \tag{9} \]

We equip \( \tilde{H}L^{1-\varepsilon}_L \) with the semi-norm
\[ \| u^\epsilon \|_{\tilde{H}L^{1-\varepsilon}_L} = \inf_{\tau \text{ satisfies}(9)} \| \tau \|_{1-\varepsilon}, \quad \epsilon \leq 0. \]

A non-homogeneous version \( \tilde{H}^{1-\varepsilon}_L \subset L^\varepsilon \cap \tilde{H}^{1-\varepsilon}_L \) can be defined using the norm \( \| u^\epsilon \|_\tau, \| u^\epsilon \|_{\tilde{H}^{1-\varepsilon}_L} \). For \( \tau > 1 \) these spaces can be identified with the usual Sobolev spaces in the Euclidean case , see \([8]\), and are part of a more general theory of Sobolev spaces on metric-measurable spaces , (see \([9]\) and \([10]\)).

Hardy-Sobolev spaces on domains in \( \mathbb{R}^n \) can be defined see \([13]\). These Hardy spaces can also be characterized, as was done in \([7]\), via a type of maximal function used by \([6]\).

We define this latter maximal function series, which we call a Sobolev sharp maximal a series functions to the case of one derivative in \( L \).

**Definition 3.2.** Let \( N(f_r) \), that \( f_r \in L^{1-\varepsilon,loc}_1 \), where \( s(B) \) is the radius of the ball \( B \), define \( Nf_r \) by
\[ \sum_r N(f_r)(x_i) = \sup_B \left( \frac{1}{s(B)} \int_B |f_r| - |f_r|_B d\mu \right). \]

The above definition is makes sense in any metric-measurable space.

**Theorem 3.3.** Let \( \mu \) be the doubling measurable and \( m \) denote metric on a metric space, cf \([11]\).
\[ H_1L^{1-\varepsilon} = \{ f_r \in L^{1-\varepsilon,loc}_1 : Nf_r \in L \}, \]
with
\[ \| f_r \|_{H_1L^{1-\varepsilon}} \sim \| Nf_r \|_1. \]

As \( f_r \in L^{1-\varepsilon,loc}_1 \) and \( N(f_r) \in L \) then \( f_r \) satisfies
\[ \sum_r f_r(x_i) - f_r(x_{i-1}) \leq C m \sum_r (x_i, x_{i-1}) (Nf_r(x_i) + Nf_r(x_{i-1})) \tag{10} \]

We have the following theorem.

**Theorem 3.4.** For \( f_r \in L^{1-\varepsilon,loc}_1 \), at every point of \( M \), that
\[ \sum_r M^+(\nabla f_r) \leq \sum_r Nf_r. \tag{11} \]

Therefore
\[ H_1L^{1-\varepsilon} \subset \check{H}L^{1-\varepsilon,\max}_L, \]
with
\[
\sum_r \| (f_r)_x \|_{\dot{H}L^{1-\epsilon}} \leq C \sum_r \| (f_r)_x \|_{\dot{H}L^1}.
\]

then
\[
\mathcal{M}^+ (\nabla f_r) \approx N f_r
\]
and
\[
\dot{H}L^{1-\epsilon} = \dot{H}L^1.
\]

IV. Proof of Theorem 3.4:
Let \( f_r \in L^{1-\epsilon,\text{loc}} \) and \( x_i \in M \). Take \( \psi \in T_1(x_i) \) as in Definition 2.7, associated to a ball containing \( x_i \).
\[
\int B f_r \div \psi \, d\mu = 0.
\]

So we can write
\[
\left| \int B f_r \div \psi \, d\mu \right| = \left| \int B \sum_r (f_r - (f_r)_B) \div \psi \, d\mu \right|
\]
\[
\leq \frac{1}{s \mu(B)} \int B \sum_r |f_r - (f_r)_B| \, d\mu \leq \sum_r N f_r (x_i).
\]

Here \( s \) is the radius of \( B \). Taking the supremum over all such \( \psi \), we get (11).

We proceed now to the proof of the reverse inequality. For this we will need the following.

Proposition 4.1. Let \( M \) is a complete Riemannian manifold satisfying (1) and (2). Let \( B \) a ball of \( M \),
\[
g_r \in L^\infty_0 (B) := \left\{ g_r \in L^\infty (B); \int B \tau d\mu = 0 \right\}.
\]

Then there exists \( \psi \in L^\infty (B, TM) \) such that \( \div \psi = g_r \).

Holds in the sense of Definition 2.2 (with \( \psi = B \)), and \( \| \psi \|_\infty \leq C \| g_r \|_\infty \).

Where \( C \) is the constant appearing in (2) and is independent of \( B \) and \( x_i \). Before proving the proposition, we conclude the proof of Theorem 4.3. Again take \( f_r \in L^{1-\epsilon,\text{loc}} \), \( x_i \in M \) and \( B \) a ball of radius \( s \) containing \( x_i \). If \( g_r \in L^\infty_0 (B) \), \( \| g_r \|_\infty \leq 1 \) and we solve \( \div \psi = g_r \) with \( \psi \) as in Proposition 4.1, then,
\[
\psi := \frac{\psi}{C \sigma \mu(B)} \in T_1(x_i),
\]

and
\[
\left| \int B \sum_r f_r g_r \, d\mu \right| = \left| \int B \sum_r f_r \div \psi \, d\mu \right| = C \sigma \mu(B) \left| \int B \sum_r (f_r)_x \div \psi \, d\mu \right|
\]
\[
\leq \frac{1}{s \mu(B)} \int B \sum_r |f_r - (f_r)_B| \, d\mu
\]
\[
= \frac{1}{s \mu(B)} \sup_{\psi \in T_1(x_i)} \left| \int B \sum_r (f_r)_x \div \psi \, d\mu \right|
\]
\[
\leq \frac{1}{s \mu(B)} \max \sup_{\psi \in T_1(x_i)} \left| \int B \sum_r (f_r)_x \div \psi \, d\mu \right|
\]
\[
= C \sum_r \mathcal{M}^+ (\nabla f_r)_x (x_i).
\]

Taking the supnorm on the left over all balls \( B \) containing \( x_i \), we get \( N(f_r)_x (x_i) \leq C \mathcal{M}^+ (\nabla f_r)_x (x_i) \).

Proof of Proposition 4.1. Let \( B \) a ball and \( \tau \in L^\infty_0 (B) \). Consider
\[
h := \{ \mathcal{H} \in \mathcal{L}(B, TM); \exists f_r \in L^{1-\epsilon,\text{loc}}(M), \mathcal{H} = \nabla f_r \text{ on } B \}.
\]

We view \( h \) as a subspace of \( \mathcal{L}(B, TM) \) with the norm
\[
\| \mathcal{H} \|_{\mathcal{L}(B, TM)} = \int_B \| h \|_x \, d\mu.
\]

Define a linear functional on \( h \) by
\[
\Lambda (\mathcal{H}) = - \int_B \sum_r g_r f_r \, d\mu \text{ if } \mathcal{H} = \nabla f_r \in h.
\]
\( \Lambda \) is well defined since \( \int_B \tau d\mu = 0 \) and is bounded on \( h \) thanks to the Poincare inequality (2).
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\[
\left| \sum \Lambda(\mathcal{H}) \right| = \int_B \sum_r g_r (f_r - (f_r)_B) \, d\mu \leq C\sigma \sum_r \|g_r\|_\infty \int_B \sum_r |\nabla f_r| \, d\mu = C\sigma \sum_r \|g_r\|_\infty \|\mathcal{H}\|_{L(B,TM)}. 
\]

The Hahn-Banach theorem shows that \( \Lambda \) can be extended to a bounded linear functional on \( \mathcal{L}(B,TM) \) with norm no greater than \( C\sigma \|g_r\|_\infty \). By duality, there exists a vector field \( \psi \in \mathcal{L}^\infty(B,TM) \) such that

\[
\int_B \sum_r \langle \psi, \nabla f_r \rangle \, d\mu = -\int_B \sum_r g_r f_r \, d\mu. 
\]

For all \( f_r \in L^{1-\varepsilon,loc}(M) \) for which \( \nabla f_r \in \mathcal{L}(B,TM) \). By Definition 2.2, this means \( \text{div} \, \psi = g_r \) on \( B \). Moreover

\[
\|\psi\|_\infty \leq C\sigma \sum_r \|g_r\|_\infty.
\]

References